## Angular momentum

Contributed by: Brian De Facio, John L. Safko
Publication year: 2014

In classical physics, the moment of linear momentum about an axis. A point particle with mass $m$ and velocity $\mathbf{v}$ has linear momentum $\mathbf{p}=m \mathbf{v}$. Let $\mathbf{r}$ be an instantaneous position vector that locates the particle from an origin on a specified axis. The angular momentum $L$ can be written as the vector cross-product in Eq. (1) The illustration shows the geometrical meaning of this equation.

$$
\begin{equation*}
\mathbf{L}=\mathbf{r} \times \mathbf{p} \tag{1}
\end{equation*}
$$

See also: CALCULUS OF VECTORS; MOMENTUM.

The time rate of change of the angular momentum is equal to the torque $\mathbf{N}$. A rigid body satisfies two independent equations of motion (the dynamical equations) given by Eqs. (2) and (3),

$$
\begin{align*}
& \frac{d}{d t} \mathbf{p}=\mathbf{F} \\
& \frac{d}{d t} \mathbf{L}=\mathbf{N} \tag{3}
\end{align*}
$$

where $\frac{d}{/ d t}$ denotes the rate of change, the derivative with respect to time $t$. Only Eq. (1) is required for a point particle. Equation (2) indicates that a rigid body acts as a point particle located at its center of mass. The motion of the center of mass depends upon the net force $\mathbf{F}$, which is the vector sum of all applied forces. Equation (3) gives the angular motion about the center of mass. The case of statics occurs when the net force and net torque both vanish. See also: statics; torque.


Geometrical definition of angular momentum. If we choose a reference point and draw a line $R$ from it to a moving body, then the line $R$ and the velocity $v$ of the body of mass $m$ define a plane. The angular momentum is a vector that lies perpendicular to this plane. If the momentum of the body is $p=m v$ and the line $R$ is a vector $r$, then the magnitude of the angular momentum $\mathbf{L}$ is $\mathrm{pr} \sin \theta$.

## Vectors and pseudovectors

Given the definition of angular momentum in Eq. (1), although it has three components, it is not a vector. The parity transformation $\pi$ transforms vectors such as $\mathbf{r}$ and $\mathbf{p}$ according to Eqs. (4).

$$
\begin{align*}
& \mathbf{1}^{\prime}=\pi \cdot \mathbf{1}^{4}=-\mathbf{1}  \tag{4a}\\
& \mathbf{D}^{\prime}=\pi \cdot \mathrm{B}=-\mathrm{D} \tag{4b}
\end{align*}
$$

Thus, $\pi \cdot \mathbf{L} \rightarrow+\mathbf{L}$, so that $\mathbf{L}$ is called a pseudovector or axial vector. Mathematically, these quantities are called tensors of odd relative weight. Examples of pseudovectors include angular momentum, torque, and magnetic fields. Two or more pseudovectors are combined by adding their components, so that the total angular momentum is the sum of the angular momenta of the individual particles.

Among the fundamental particles, the photon field is a massless vector field, the rho ( $\rho$ ) meson is a vector field, and the $a_{1}$ meson is a pseudovector field. See also: elementary particle; parity (Quantum mechanics).

## Rigid-body motion

To study the nonrelativistic mechanics of a rigid body made up of $N$ particles, it is convenient to write the instantaneous velocity $\mathbf{v}_{\alpha}$ of the $\alpha$-th particle in terms of angular velocity $\omega$ as $\mathbf{v}_{\alpha}=\omega \times \mathbf{r}_{\alpha}$. For this system, Eq. (1) becomes Eq. (5),

$$
\begin{equation*}
\mathbf{L}=\sum_{\alpha=1}^{N} m_{\alpha} \mathbf{r}_{\alpha} \times\left(\omega \times \mathbf{r}_{\alpha}\right)=\mathbf{I} \cdot \omega \tag{5}
\end{equation*}
$$

where $I$ is the moment of inertia tensor. The moment of inertia tensor can be written in matrix form as Eq. (6). Representative components of this tensor are given by Eqs. (7).

$$
\begin{gather*}
I_{i j}=\sum_{\alpha=1}^{N} m_{\alpha}\left[r_{\alpha}^{2} \delta_{i j}-\left(r_{\alpha}\right)_{i}\left(r_{\alpha}\right)_{j}\right] \\
I_{11}=\sum_{\alpha=1}^{N} m_{\alpha}\left(y_{\alpha}^{2}+z_{\alpha}^{2}\right)  \tag{7a}\\
I_{12}=-\sum_{\alpha}^{N} m_{\alpha} x_{\alpha} y_{\alpha} \tag{7b}
\end{gather*}
$$

Since $\mathbf{I}$ is symmetric, it can always be diagonalized and expressed in principal axis (or symmetry axis) form, $\mathbf{I}_{P A}=$ diag $\left(I_{1}, I_{2}, I_{3}\right)$, and in this form it is constant in time: $\dot{\mathbf{I}}_{P A}=0$. The Euler angle coordinates $(\alpha, \beta, \gamma)$ are useful because they can be used to rotate a rigid body into its principal axis frame. See also: euler angles; moment of INERTIA; ROTATIONAL MOTION.

When all of the forces that act on a system are conservative (that is, $\nabla \times \mathbf{F} \equiv \mathbf{O}$ ), a potential energy function $V(\mathbf{r})$ exists for which $\mathbf{F}=-\nabla V(\mathbf{r})$. Consider a rigid body of mass $M$ and inertia tensor $\mathbf{I}$ subject to two conservative forces $\mathbf{F}_{1}=-\nabla_{R} V_{1}(\mathbf{R})$ and $\mathbf{F}_{2}=-\nabla_{\alpha 0} V_{2}\left(\alpha_{0}\right)$, where $\mathbf{R}$ is the position vector of the center of mass, and $\alpha_{0}=(\alpha$, $\beta, \gamma)$ are moment-arm-weighted Euler angles. The quantities $\nabla_{R}$ and $\nabla_{\alpha 0}$ are gradient operators. The equations of
motion (2) and (3) become Eqs. (8) and (9).

$$
\begin{equation*}
M \mathbf{R}=\mathbf{F}_{1} \tag{8}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{I} \cdot \dot{\omega}=\mathbf{N}_{2}=\mathbf{r} \times \mathbf{F}_{2} \tag{9}
\end{equation*}
$$

Using $\mathbf{I}_{P A}$ and writing out Eq. (9) in component form in the rotating principal-axis frame yields Eqs. (10).

$$
\begin{align*}
& I_{1} \omega_{x}-\left(I_{2}-I_{3}\right) \omega_{y} \omega_{z}=N_{x} \\
& I_{2} \omega_{y}-\left(I_{3}-I_{1}\right) \omega_{x} \omega_{z}=N_{y}  \tag{10b}\\
& I_{3} \omega_{z}-\left(I_{1}-I_{2}\right) \omega_{x} \omega_{y}=N_{z} \tag{10c}
\end{align*}
$$

See also: Energy; RIGID-body dynamics.

## Conservation

A symmetry is a transformation that leaves a physical system unchanged. A physical quantity is called invariant under a transformation if it remains the same after being transformed. For example, the solutions to Eqs. (2) and (3), or equivalently Eqs. (8) and (9), are invariant under change of the coordinate origin or orientation of the $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ axes. The freedom to choose any orientation of coordinate axis is called rotational invariance, because one choice of axes can be rotated into another.

In physics, the rotational invariance follows from the isotropy and homogeneity of space that has been experimentally established to high accuracy. The diagonalization transformation $\mathbf{I} \rightarrow \mathbf{I}_{P A}$ is carried out by rotating the axes, and thus is a consequence of rotational invariance. The principal axes of a rigid body are called the symmetry axes because their mechanical description is simplest there. (Rotational invariance says that the mechanics can be solved in any orientation of the axes but not that an arbitrary orientation is simplest.)

The study of symmetry shows that one of the deepest relations in physics is that between dynamics and conservation. A physical quantity $B$ is conserved if Eq. (11)

is satisfied; that is, $B$ is constant in time although it may vary in space. Noether's theorem states that if a physical system is invariant under a continuous symmetry, a conservation law exists, provided that the observable in question decreases rapidly enough at infinity. Thus, when the force is zero everywhere (the system is invariant under translation in space), the linear momentum is conserved. If the torque is zero everywhere (the system is invariant under rotation), the angular momentum is conserved. If the system is invariant under translations in time, the total energy is conserved. The converse to Noether's theorem is false, as conserved quantities do not imply continuous symmetries. For example, the parity transformation $\pi$ of Eqs. (4) can be conserved but is not continuous.

The set of all three-dimensional rotations form a group called $S O(3)$ : S for special (det $=+1$ ), O for orthogonal (length-preserving), and 3 for the space dimension. The set of $2 \times 2$ complex, unitary (length-preserving) special transformations is called $\mathrm{SU}(2)$. There is a $2: 1$ homorphism of $\mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$, and $\mathrm{SU}(2)$ is called the covering group of $\mathrm{SO}(3)$.

Angular momentum is important in the evolution of celestial objects. The shapes of these bodies and collections of these bodies, spiral galaxies for instance, follow from the space-time variation of their torques. There is a puzzling problem along these lines: If the universe can rotate, it is not clear why it rotates so slowly; that is, why the anisotropics are so small. A proposed solution is based on the hypothesis of an inflationary epoch in the very early universe, about $10^{-35}$ s after the big bang. See also: big bang theory; Conservation of energy; conservation OF MOMENTUM; COSMOLOGY; SYMMETRY LAWS (PHYSICS); UNIVERSE.

## Quantum angular momentum

Quantum mechanics has a richer and more complicated structure than classical physics. Because of this, the relationship between symmetry and conservation is even more useful. Whereas in classical physics the observables and states coincide, in quantum mechanics the states of the system correspond to probability amplitudes $\psi$ and the observables $\mathbf{A}$ to self-adjoint operators (hermitian operators with suitable boundary conditions). According to the Heisenberg uncertainty principle, all observables cannot be simultaneously known, rather only those that commute. Two commuting self-adjoint operators can be diagonalized by a single unitary transformation. The complete set of commuting observables specifies the maximum amount of sharp information possible about a quantal system. The time dependence of a state, represented in $\mathbf{x}$-space, where $\mathbf{x}$ is the position
coordinate, as $\psi(\mathbf{x}, t)$, is given by the Schrödinger equation (12),

$$
\begin{equation*}
\hat{H} \psi(\mathbf{x}, t)=i \hbar\left(\frac{\partial \psi}{\partial t}\right)(\mathbf{x}, t) \tag{12}
\end{equation*}
$$

where $\hat{\mathrm{H}}$ is the system hamiltonian (energy) operator and $\hbar=1.05 \times 10^{-34}$ joule-second and is Planck's constant divided by $2 \pi$. See also: nonrelativistic quantum theory; quantum mechanics.

Rotational invariance under $\mathrm{SO}(3)$ transformations in quantum mechanics implies the angular momentum commutation relations for a single particle given in vector form by Eq. (13).

$$
\begin{equation*}
\mathbf{J} \times \mathbf{J}=i \hbar \mathbf{J} \tag{13}
\end{equation*}
$$

The $z$ component of this equation can be written as Eq. (14),

$$
\begin{equation*}
\left[J_{x}, J_{y}\right]=i \hbar J_{z} \tag{14}
\end{equation*}
$$

and the $x$ and $y$ components are expressed in similar fashion, where $[A, B]$ is the commutator of $A$ and $B$, defined by Eq. (15).

$$
\begin{equation*}
[A, B]=A B-B A \tag{15}
\end{equation*}
$$

The quantum-mechanical momentum $\mathbf{J}$ is the most general observable satisfying Eq. (13). The orbital angular momentum $\mathbf{L}=i \hbar \mathbf{r} \times \nabla$, where $\nabla$ is the gradient operator, is a special case of a quantum-angular momentum. The total angular momentum $\mathbf{J}=J_{x} \mathbf{i}+J_{y} \mathbf{j}+J_{z} \mathbf{k}$ has squared length $j^{2}=J_{x}{ }^{2}+J_{y}{ }^{2}+J_{z}{ }^{2}$. It is straightforward to use the operator identity $\left[A^{2}, B\right]=A[A, B]+[A, B] A$ together with Eq. (13) to show that Eq. (16)

$$
\begin{equation*}
\left[J^{2}, J_{i}\right]=0 \tag{16}
\end{equation*}
$$

is satisfied for $i=x, y, z$. An inspection of Eqs. (13) and (16) shows that the angular momentum complete set of commuting observables is $\left\{j^{2}, J_{i}\right\}$, where any single component $J_{i}$ can be simultaneously diagonal with $j^{2}$. Conventionally $J_{z}$ is chosen as the diagonal component, and $j^{2}$ is called the Casimir operator of $\mathrm{SO}(3)$. The two nonself-adjoint ladder operators $J_{ \pm}=J_{x} \pm J_{y}$ are useful for determining the properties of angular momentum states.

Let $\psi(J, M, r)$ be a simultaneous eigenfunction of the operators $j^{2}$ and $J_{z}$ with eigenvalues $\lambda$ and $\mu$, so that Eqs. (17) and (18)

$$
\begin{align*}
& J^{2} \psi(J, M, r)=\lambda \psi(J, M, r)  \tag{17}\\
& J_{z} \psi(J, M, r)=\mu \psi(J, M, r) \tag{18}
\end{align*}
$$

are satisfied. The variable $r$ in $\psi$ stands for all other observables which commute with $j^{2}$ and $J_{z}$. Repeated use of the angular momentum commutation relations of the operators $J_{+}$and $j^{2}$ proves that $\lambda=J(J+1) \hbar^{2}$ and that $\mu=$ $M \hbar$, where $M=-J,-J+1, \ldots, J-1, J$ for each value of $J$. Further, $J=n / 2$, where $n$ is a positive integer, so that $0,1 / 2, \frac{3}{\sqrt{2}}, 3 / 2,1, \frac{5}{\sqrt{2}}, 5 / 2, \ldots$ are the allowed values of total angular momentum. The eigenfunctions for half-odd-integer values of total angular momentum are called spinors. A value $S$ of spin, together with $S_{z}$, labels intrinsic spin states. Spinors are new features of rotational invariance. They have the surprising property of changing sign under an $\mathrm{SO}(3)$ rotation through $2 \pi$ radians, $\mathbf{R}(2 \pi)$; that is, Eq. (19) is satisfied.

$$
\begin{equation*}
\mathbf{R}(2 \pi) \psi(J, M, r)=-\psi(J, M, r) \tag{19}
\end{equation*}
$$

The states $\psi(J, M, r)$ can be spinor-valued since measured probability densities depend only upon $|\psi(J, M, r)|^{2}$ $d V$ (where $d V$ is a volume element), whereas the observables must be tensors to have sensible classical limits. The doublet Zeeman splitting of an unpaired electron in an external magnetic field is an indication that the electron has this strange internal structure. Thus, $\mathbf{J}=\mathbf{L}+\mathbf{S}$ is the total angular momentum. See also: spin (QUANTUM MECHANICS).

## Quantum addition of angular momenta

The discussion above gives the properties of a single angular momentum. Two angular momenta, $\mathbf{J}_{1}$ and $\mathbf{J}_{2}$, are called independent if all components of one commute with all components of the other. From the preceding discussion, the sets of eigenfunctions $\psi\left(J_{1}, M_{1}, r\right)$ and $\psi\left(J_{2}, M_{2}, r\right)$ can be determined. Let $\mathbf{J}=\mathbf{J}_{1}+\mathbf{J}_{2}$. Then the eigenfunction $\psi(J, M, r)$ can be expressed as a linear combination of the $\psi\left(J_{1}, M_{1}, R\right)$ and $\psi\left(J_{2}, M_{2}, r\right)$ by Eq. (20),

$$
\begin{array}{r}
\psi(J, M, r)=\sum \mathrm{C}\left(J_{1} J_{2} J, M_{1} M_{2}\right) \psi\left(J_{1}, M_{1}, r\right)  \tag{20}\\
\cdot \psi\left(J_{2}, M_{2}, r\right)
\end{array}
$$

where $\mathrm{C}\left(J_{1} J_{2} J, M_{1} M_{2}\right)$ is the Wigner coefficient (often erroneously called the Clebsch-Gordon coefficient). The summation is over values of $M_{1}$ and $M_{2}$ such that $M=M_{1}+M_{2}$, and $J$ must have one of the set of values $J=J_{1}+$ $J_{2}, J_{1}+J_{2}-1$.

The Wigner coefficients play another important role in the Wigner-Eckart theorem. Let T( JM ) be a family of tensor operators with angular momentum $J M$, and consider the quantum matrix elements $j^{\prime} m^{\prime}|\mathbf{T}(J M)| j m$. These matrix elements represent transition probability amplitudes from quantum state $j m$ to state $j m^{\prime}$ caused by $\mathbf{T}(J M)$. The Wigner-Eckart theorem states that these matrix elements can be expressed by Eq. (21),

$$
\begin{equation*}
\left\langle j^{\prime} m\right| \mathbf{T}(J M)|j m\rangle=\mathrm{C}\left(j J j^{\prime}, m M\right)\left\langle j^{\prime}\right| \mathbf{T}(J)|j\rangle \tag{21}
\end{equation*}
$$

where $j^{\prime}|\mathbf{T}(J)| j$ ? is the reduced matrix element and is independent of $m M m^{\prime}$. The angular momentum conservation is contained in the Wigner coefficient; the details of the particular operator $\mathbf{T}$ are contained only in the reduced matrix element.

The operator $\mathbf{T}$ in atomic spectroscopy is a perturbation hamiltonian from an incident electromagnetic wave that drives atomic transitions. Thus, the Wigner-Eckart theorem, through the Wigner coefficients, determines which transitions vanish, that is, the selection rules governing the process. Much of atomic spectroscopy follows from the rotational symmetry, or the quantum theory of angular momentum. See also: atomic structure and spectra; SELECTION RULES (PHYSICS).

Brian De Facio, John L. Safko

## Bibliography

L. C. Biedenharn and J. D. Louck, Angular Momentum in Quantum Pbysics: Theory and Application, Cambridge University Press, Cambridge, UK, 1981, reissue 2009
E. U. Condon and H. Odabasi, Atomic Structure, Cambridge University Press, Cambridge, UK, 1980, reissue 2010
E. P. Wigner, Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra, Academic Press, New York, 1959

## Additional Readings

N. J. Kasdin and D. A. Paley, Engineering Dynamics: A Comprebensive Introduction, Princeton University Press, Princeton, NJ, 2011

