Wave motion

Contributed by: Leo L. Beranek

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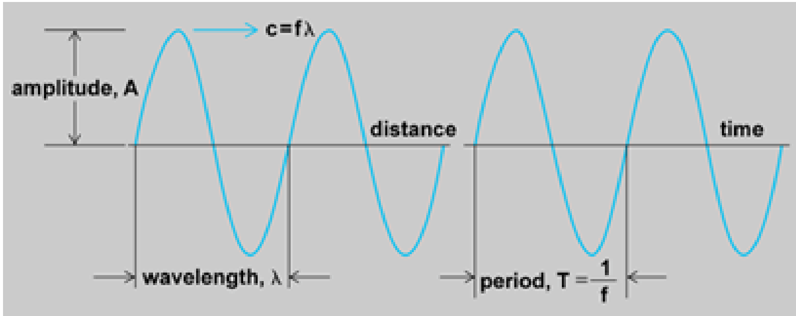
The process by which a disturbance at one point in space is propagated to another point more remote from the source with no net transport of the material of the medium itself. For example, sound is a form of wave motion; wind is not. Wave motion can occur only in a medium in which energy can be stored in both kinetic and potential form. In a mechanical medium, kinetic energy results from inertia and is stored in the velocity of the molecules, while potential energy results from elasticity and is stored in the displacement of the molecules.

In a free traveling wave (as distinguished from a stationary or standing wave) one part of the medium disturbs an adjacent part, thereby imparting energy to it. This portion of the medium, in turn, disturbs another part, thereby causing a flow of energy in a given direction away from the source (see **illustration**). More technically, wave propagation is the result of kinetic energy at one point being transferred into potential energy at an adjacent point, and vice versa. The rate of travel of the disturbance, or velocity of propagation, is determined by the constants of the medium. A stationary wave is the combination of two waves of the same frequency and strength traveling in opposite directions so that no net transfer of energy away from the source takes place. A standing wave is the same but with the returning wave (toward the source) being of lesser intensity than the outwardly traveling wave so that a net transfer of energy away from the source does take place.

Wave motion can occur in a vacuum (electromagnetic waves), in gases (sound waves), in liquids (hydrodynamic waves), and in solids (vibration waves). Electromagnetic waves can also travel in gases, liquids, and solids provided that the electrical conductivity of the medium is not perfect or that the imaginary part of the dielectric constant is not infinitely great. By current usage, elastic waves propagated in gases, liquids, and solids, regardless of whether one can hear them or not, are called acoustic waves.

In this article, it is shown that wave motion can exist in any continuous medium in which potential and kinetic energy can be stored. The wave equation is derived, and solutions for electromagnetic and mechanical media are given. It is shown that longitudinal waves may exist in all mechanical media, and that transverse waves may exist in electromagnetic and solid mechanical media. In all cases, viscosity or heat losses in mechanical media, and resistance and magnetic losses in electromagnetic media are neglected. Only homogeneous and isotropic media are considered. For information related to the present discussion *See also:* DIFFRACTION; ELECTROMAGNETIC RADIATION; HARMONIC MOTION; HUYGENS' PRINCIPLE; INTERFERENCE OF WAVES; LIGHT; MAXWELL'S EQUATIONS; REFRACTION OF WAVES; SOUND; SUPERPOSITION PRINCIPLE; VIBRATION; WAVE EQUATION; SURFACE WAVES.



Relation between frequency, wavelength, and velocity in wave propagation.

Fundamental relations

A wave is commonly referred to in terms of either its wavelength or its frequency. In any type of wave motion, these two quantities are related to a third quantity, velocity of propagation, by Eq. (1),

$$f\lambda = c \tag{1}$$

where f = frequency, λ = wavelength, and c = velocity of propagation. The period T is the reciprocal of the frequency, and the amplitude A is the maximum magnitude taken on by the variable of the wave at a given point in space. The physical significance of these quantities is presented in the illustration. It is a basic property of wave motion that the frequency of a wave remains constant under all circumstances except for a relative motion between the source of the wave and the observer. The case of a frequency shift due to relative motion, known as the Doppler effect, has many interesting applications. On the other hand, the velocity of propagation is dependent on the properties of the medium (and sometimes also on the frequency), and the wavelength will vary with the velocity in accordance with Eq. (1). *See also:* DOPPLER EFFECT.

Electromagnetic waves

The media in which electromagnetic waves travel possess no elasticity or inertia, but the ability to store energy in the electric and magnetic fields. The electric field corresponds in every respect to the field of an irrotational fluid motion, and the mathematical formulations of the motions of acoustic waves and electromagnetic waves are similar.

In the customary manner, in an electromagnetic medium one defines the vectorial factors **E** as the electric field strength and **H** as the magnetic field strength, with both magnitudes and directions at every point in space.

Next, it is assumed that the medium is homogeneous and isotropic (that is, that the dielectric constant κ and the permeability μ are constants and scalars), that there are no applied electromotive forces in the portion of the medium being dealt with, and that the electrical conductivity σ of the medium is zero. (Waves can be propagated in media where σ is greater than zero but less than infinity.) Then the field equations can be written as Eqs. (2)–(5),

$$\frac{\kappa}{c}\frac{\partial \mathbf{E}}{\partial t} = \mathrm{curl}\ \mathbf{H} \tag{2}$$

$$-\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t} = \mathrm{curl}\ \mathbf{E} \tag{3}$$

$$\mathrm{div}\ \mathbf{H} = 0 \tag{4}$$

$$\mathrm{div}\ \mathbf{E} = 0 \tag{5}$$

where c will be shown to be the velocity of propagation of the electromagnetic wave in a vacuum, where $\kappa = \mu = 1$. *See also:* CALCULUS OF VECTORS.

Wave equation. J. C. Maxwell recognized about 1863 that these basic equations could be combined to yield an equation resembling the wave equation for mechanical wave motion. Thus he predicted the existence of electromagnetic waves which had not been suspected before. Later, electromagnetic waves proved to be identical with light waves.

The combination of Eqs. (2)–(5) yields the wave equation (6).

$$\frac{\kappa\mu}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E} \tag{6}$$

In a vacuum, the wave equation becomes Eq. (7) and also Eq. (8),

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E} \tag{7}$$

$$\frac{\partial^2 \mathbf{H}}{\partial t^2} = c^2 \nabla^2 \mathbf{H} \tag{8}$$

where ∇^2 is a scalar operator called the laplacian. *See also:* LAPLACIAN.

Plane wave propagation. For simplicity, particular solutions of the wave equation for the case of plane wave propagation will now be sought. A wave is called plane when a family of parallel planes can be taken in the field such that the electric and magnetic field strengths are constant in magnitude and direction at all points of any given member of the family; the planes are called wavefronts, the direction perpendicular to them the wave normal. If the axis of x is taken in the direction of the wave normal, then the wavefronts are parallel to the plane of yz.

Since, for this case, **E** and **H** are to be constant in any one wavefront, the partial derivatives with respect to y or z must vanish. Then the field equations reveal that Eq. (9) is valid.

$$\frac{\partial E_x}{\partial t} = \frac{\partial E_x}{\partial x} = \frac{\partial H_x}{\partial t} = \frac{\partial H_x}{\partial x} = 0 \tag{9}$$

Therefore, insofar as wave propagation is concerned, Eq. (10) holds.

$$E_x = H_x = 0 \tag{10}$$

The meaning of Eq. (10) is that neither **E** nor **H** can have a periodically changing component in the direction in which the wave is traveling. That is to say, the waves are not longitudinal but transverse.

Of the remaining four equations developing out of the field equations, two of them connect E_y and H_z, and the other two E_z and H_y. Dealing with the pairs, one obtains the one-dimensional wave equation (11),

$$\frac{\partial^2(\ )}{\partial t^2} = c^2 \frac{\partial^2(\ )}{\partial x^2} \tag{11}$$

where E_y, E_z, H_y, or H_z may be inserted within the parentheses.

The general solution of Eq. (11) can be written in the form of Eq. (12).

$$E_y = F_1(x - ct) + F_2(x + ct) \tag{12}$$

is The first term of Eq. (12) associated with a wave traveling in the positive x direction, and the second term with a wave traveling in the minus x direction. This is true because the argument of F_1 is the same whenever $x = ct$, so that as t becomes greater, the position of a wavefront moves in the positive direction of x. Since for F_2, $-x = ct$, the opposite is true.

If only the outward traveling wave is considered, Eq. (13)

$$E_y = F_1(x - ct) \tag{13}$$

holds. Such a wave is transmitted without change of shape, because its position x can always equal ct. That is, at any given elapsed time t, E_y will have the same value at $x = ct$ as it had at $x = t = 0$. Also, because $c = \frac{x}{t}$, c has the dimensions of a velocity and is the speed at which the wave travels.

Light is also an electromagnetic process so that one should be able to state the optical properties of a substance once its electrical constants are given. However, at the high frequencies of visible light, there is generally dispersion in the medium so that κ and μ vary with frequency, and this variation must be taken into account if Maxwell's equations are to give a description of optical phenomena which fits measured data.

Acoustic waves in gases

In gases, the existence of inertia (resulting in kinetic energy) and elasticity (resulting in potential energy) is obvious. For example, imagine taking from the medium a small packet of gas enclosed by a weightless deformable membrane. When this packet is squeezed, it is found to have stiffness (or elasticity). Its mass equals the mass of the air molecules contained therein. Also, the deformable membrane keeps the mass inside constant.

From this description, it can be seen that it is a relatively simple matter to describe the motion of such a packet as a function of time. The mass element is labeled in any convenient manner, the most common being by its location at any convenient time. This method is called the lagrangian description of motion.

To derive the equations for wave motion in a gas, one expresses the concepts of inertia by Newton's second law, of elasticity by the perfect gas law, and of the deformable packet by the conservation of mass law. The packet is assumed to be rectangular in shape. To find the equation of motion, the supposition is made that the box is situated in a medium in which the pressure p changes in space at a space rate given by Eq. (14),

$$\text{grad } p \equiv \mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y} + \mathbf{k}\frac{\partial p}{\partial z} \tag{14}$$

where **i**, **j**, and **k** are unit vectors in the x, y, and z directions, respectively, and p is the pressure at a point.

The net force **f** acting to move the box in some direction is equal to the vector summation of the gradients in force across the three pairs of faces of the packet times the respective separations of these faces; in the positive direction, Eq. (15)

$$\mathbf{f} = -\left[\mathbf{i}\left(\frac{\partial p}{\partial x}\Delta x\right)\Delta y\Delta z + \mathbf{j}\left(\frac{\partial p}{\partial y}\Delta y\right)\Delta x\Delta z + \mathbf{k}\left(\frac{\partial p}{\partial z}\Delta z\right)\Delta x\Delta y\right] = -V\text{grad } p \tag{15}$$

holds. Here V is the the average volume of the packet. The positive gradient causes an acceleration of the box in the negative direction of x.

By Newton's second law, the force acting to accelerate the packet is given by Eq. (16),

$$\mathbf{f} = \rho' V\frac{\partial \mathbf{q}}{\partial t} \tag{16}$$

where **q** is the average vector velocity of the gas in the packet, ρ' the average density of the gas in the packet, and $\rho' V$ the total (constant) mass of the gas in the packet. In writing Eq. (16), it has been assumed that the packet is never displaced from its equilibrium position by a significant part of a wavelength of sound.

Combining Eqs. (15) and (16) yields Eq. (17).

$$-\text{grad } p = \rho_0\frac{\partial \mathbf{q}}{\partial t} \tag{17}$$

In keeping with the approximation just stated, it has been assumed that ρ' (the instantaneous density) does not appreciably deviate from the average density ρ_0. These approximations are acceptable, providing the sound pressures are below about 100 dynes/cm^2 (10 newtons/m^2).

Assuming adiabatic expansions and contractions of the gas, and that the gas is perfect, Eq. (18) is obtained.

$$\frac{dP}{P} = -\frac{\gamma dV}{V} \tag{18}$$

Here *P* is the total pressure in the gas and γ is the ratio of specific heat at constant pressure to specific heat at constant volume.

P and *V* can be defined by Eqs. (19),

$$\begin{aligned} P &= P_0 + \\ V &= V_0 + \tau \end{aligned} \tag{19}$$

where p and τ are time-varying quantities, and P_0 and V_0 are equilibrium values. If inequalities (20) hold,

$$\begin{aligned} p &\ll P_0 \\ \tau &\ll V_0 \end{aligned} \tag{20}$$

Eq. (21) is valid.

$$\frac{1}{P_0}\frac{\partial p}{\partial t} = \frac{-\gamma}{V_0}\frac{\partial \tau}{\partial t} \tag{21}$$

To satisfy the law of mass conservation, one writes Eq. (22)

$$\tau = V_0 \operatorname{div} \xi \tag{22}$$

where ξ is the average vector displacement of the box.

Differentiation of Eq. (22) with respect to time, and substitution of it in Eq. (21), yields Eq. (23).

$$\frac{\partial p}{\partial t} = -\gamma P_0 \operatorname{div} \mathbf{q} \tag{23}$$

The elimination of p from Eqs. (17) and (23) yields the wave equation (24),

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p \tag{24}$$

where by definition Eq. (25) applies.

$$c^2 = \frac{\gamma P_0}{\rho_0} \tag{25}$$

One-dimensional plane waves. An acoustic wave is called plane when a family of parallel planes can be taken in the medium such that the pressures and particle velocities are constant in magnitude and the particle velocities are constant in direction at all points of any given member of the family.

Since p and $\mathbf{q}$ are to be constant in any one wavefront, the partial derivatives with respect to y or z must vanish, and in Eq. (23), Eq. (26) applies.

$$\operatorname{div} \mathbf{q} = \frac{\partial q_x}{\partial x} \tag{26}$$

Since Eq. (26) reveals that the x component of $\mathbf{q}$ is the only component of $\mathbf{q}$ remaining in the equations basic to the wave equation, the wave is longitudinal.

The one-dimensional wave equation becomes Eq. (27).

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2} \tag{27}$$

The general solution to Eq. (27) is exactly the same as that for electromagnetic waves and is given by Eq. (12). The discussion following Eq. (12) is also valid here.

One-dimensional spherical waves. In free space, it is frequently desired to express mathematically the radiation of sound from a nondirectional source. In this case the sound wave expands as it travels away from the source and the wavefront is always a spherical surface. The operator on the right side of Eq. (24) can be written in a form suitable to spherical coordinates.

Assuming equal radiation in all directions, the wave equation in one-dimensional spherical coordinates then becomes Eq. (28).

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \quad (28)$$

Differentiation shows that Eq. (28) can also be written as Eq. (29).

$$\frac{\partial^2 (pr)}{\partial t^2} = c^2 \frac{\partial^2 (pr)}{\partial r^2} \quad (29)$$

Equation (29) has the same form as that of Eq. (27). Hence, the same formal solution applies to either equation, except that the dependent variable is $p(x,t)$ in one case and $pr(r, t)$ in the other case.

The solution to Eq. (29) for the outward traveling wave only (free space) is thus given by Eq. (30).

$$p = \frac{1}{r} F_1(r - ct) \quad (30)$$

Note that, just as for the plane wave, the wave is propagated without change of shape. However, the magnitude of the sound pressure decreases inversely with distance owing to the spreading of the wave as it propagates.

Acoustic waves in liquids

Acoustic waves in liquids obey the same equations as those in gases. The velocity of propagation is greater in liquids than in gases, and viscous losses are often higher. For information on an important example of wave motion in liquids *See also:* UNDERWATER SOUND.

Elastic and flexural waves in solids

Several different types of acoustic waves may exist in solids, depending on the different manners in which potential energy is stored in the solid. The wave equations associated with several of these types are reviewed in the following paragraphs.

Waves on flexible stretched strings. The wave equation for a flexible stretched string is given by Eq. (31),

$$\frac{\partial^2 \xi_y}{\partial t^2} = c^2 \frac{\partial^2 \xi_y}{\partial x^2} \tag{31}$$

where ξ_y is the displacement of the string at a point x along the string in the y direction (perpendicular to the string). The speed of propagation c is equal to the square root of the ratio of the tension (in dynes) to the linear density of the string (in g/cm^2).

The solution to this equation is identical to that for Eq. (11). Because the motion of the elements of the string is perpendicular to the string, at least for small displacements, the waves are said to be transverse.

Flexural waves in bars. For the purposes of this article, it is assumed that a string has tension with negligible stiffness, while a bar has stiffness without tension. When a bar is bent, its lower half is compressed and its upper half is stretched, or vice versa. When the bending force is removed, the bar attempts to regain its equilibrium position. The restoring force is due to the moment of the forces about the neutral plane in the bar and is related to the cross-sectional dimensions and the Young's modulus of the material.

The transverse wave equation describing the motion of such a bar is Eq. (32),

$$\frac{\partial^2 \xi_y}{\partial t^2} = -\kappa^2 \frac{Y}{\rho} \frac{\partial^4 \xi_y}{\partial x^4} \tag{32}$$

where ξ_y is the displacement perpendicular to the neutral plane of the bar, Y is Young's modulus, p is the density of the bar, and κ is the radius of gyration of the cross section. Values of κ for some of the simpler cross-sectional shapes are given by Eqs. (33).

$$\kappa = \frac{a}{\sqrt{12}} \tag{33a}$$

Rectangle: Length *b* parallel to center line, width *a* perpendicular to center line, Circle: Radius *a*,

$$\kappa = \frac{a}{2} \tag{33b}$$

Circular ring: Outer radius *a*, inner radius *b*,

$$\kappa = 0.5(a^2 + b^2)^{1/2} \tag{33c}$$

Equation (32) differs from the usual wave equation (11) in that it has a fourth derivative with respect to *x* instead of a second derivative.

The function $F_1(x - ct)$ is not a solution, so that a bar satisfying Eq. (32) cannot have waves traveling along it with a velocity independent of frequency and with an unchanged shape.

If one considers excitation of the bar by one frequency *f* at a time, a solution to Eq. (32) may be written as Eq. (34),

$$\xi_y = A\cos\left[2\pi(\mu x - ft) + \phi\right] \tag{34}$$

where $\mu = (\frac{f^2\rho}{4\pi^2 Y\kappa^2})^{\frac{1}{4}}$, $f = 2\pi\mu^2\kappa\sqrt{\frac{Y}{\rho}}$, A = amplitude, and φ = phase angle.

The velocity of propagation of the wave is equal to $(\frac{f}{\mu}) = (\frac{4\pi^2 Y\kappa^2}{\rho})^{\frac{1}{4}}\sqrt{f}$. It obviously depends on the frequency *f* of the exciting wave. Such a velocity for a simple harmonic wave is called the phase velocity. For a complex wave with several components differing in frequency, the shape must change with distance of travel. A bar is sometimes said to be a dispersing medium for waves of bending. *See also:* PHASE VELOCITY.

There is also a possibility for a longitudinal wave in such a bar. In this case the bar is excited at an end in the direction of its longitudinal axis. The wave equation is identical to Eq. (31), except that ξ_x replaces ξ_y. The motions of the particles of the bar are in a line with the longitudinal axis of the bar. Longitudinal and transverse waves may, and generally do, coexist.

Waves in plates. Wave motion in a plate is similar to that in a bar. The bending of a plate compresses the material on the inside of the bend and stretches it on the outside. But when a material is compressed, it tries to spread out in a direction perpendicular to the compressed force, so that when a plate is bent downward in one direction there is a tendency for it to curl up in a direction at right angles to the bend. The ratio of the sideways spreading to the compression is called Poisson's ratio and is designated by the letter σ.

The wave equation for the plate is given by Eq. (35),

$$\Delta^4\eta + \frac{12}{(bc_L)^2}\frac{\partial^2\eta}{\partial t^2} = 0 \qquad (35)$$

where η is the displacement of the plate perpendicular to its surface, b is the thickness of the plate, and c_L is the longitudinal plate velocity given by Eq. (36).

$$c_L = \sqrt{\frac{Y}{\rho(1-\sigma^2)}} \qquad (36)$$

From a solution to the equation it is possible to find the velocity of propagation for the transverse (bending) wave, as in Eq. (37).

$$c_B = \sqrt{1.8bc_L}\sqrt{f} \qquad (37)$$

Just as for the bar, the velocity of propagation is dependent on frequency, and the plate is said to be a dispersing medium.

Steady-state wave motion

Very frequently the source of a wave is steady and the wave produced by it is periodic, at least for a long period of time compared to the buildup and decay times of the wave. When this is true, it is convenient to specify the functions $F_1(x - ct)$ and $F_2(x + ct)$ of Eq. (12) by a summation of sinusoidal functions, as in Eq. (38),

$$F(x-ct) = \sum_{\nu} A_\nu \cos\left[\omega_\nu\left(t - \frac{x}{c}\right) + \theta_\nu\right] \qquad (38)$$

where $\omega_\nu = 2\pi f_\nu$;f_ν is the frequency of the νth component of the wave, and θ_ν is the phase angle of the νth component, which is determined when $x = t = 0$.

From the well-known theory of Fourier series, any function $F_1(x - ct)$, if it repeats on itself periodically, can be represented by a linear summation of sine or cosine wave functions. The component with the lowest frequency f_1 is the fundamental component or the first harmonic; each higher component, called the $(\nu - 1)$th overtone, or the νth harmonic, has a frequency equal to νf_1. In other words, ν is an integer, and the waveform $F_1(x - ct)$ is represented by a linear summation of cosine terms with frequency components f_ν that are harmonically related (f, $2f$, $3f$, $4f$, and so forth) and at $t = x = 0$ have phase angles θ_1, θ_2, θ_3, and so forth, that may differ from zero. *See also:* FOURIER SERIES AND TRANSFORMS.

As a further simplification, each component of Eq. (38) is usually written as Eq. (39),

$$P = A\cos(\omega t - kx + \theta) \tag{39}$$

where $k = \frac{\omega}{c}$ is called the wave number. The period of this component is T and equals $\frac{1}{f}$. As before, θ is the phase angle in radians.

Note that A is the peak amplitude of the component wave. Generally, in acoustics, measuring devices read the root-mean-square value of a wave, so that the intensity of the wave is designated by $A_{\text{rms}} = \frac{A}{\sqrt{2}}$, and the strength of a wave represented by Eq. (38) is given by Eq. (40).

$$A_{\text{rms}} = \sqrt{A_{1_{\text{rms}}}^2 + A_{2_{\text{rms}}}^2 + \cdots} \tag{40}$$

See also: ROOT-MEAN-SQUARE.

Leo L. Beranek

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Additional Readings

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